Int. J. Solids Structures, 1969, Vol. 5, pp. 893 to 914. Pergamon Press. Printed in Great Britain

MEAN WAVES IN LAMINATED RANDOM MEDIA†

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Abstract—Stochastic displacement equations of laminated random media are obtained by employing a simple version of the so-called "effective stiffness" theories of laminated media. A perturbation procedure leads to deterministic equations for mean wave propagation. Special attention is given to uncoupled modes of mean waves propagating perpendicular or parallel to the direction of the layering, and dispersion relations for those plane wave trains are established.

1. INTRODUCTION

IN RECENT years a dynamical continuum theory for layered media has been developed by Herrmann, Achenbach, and coworkers [1, 2]. This theory, which approximately describes the dynamic behavior of laminated composites for small amplitude elastic waves, has been worked out in detail for a medium of alternating layers of two homogeneous isotropic materials. It has been shown [2] that the dispersion of plane harmonic free waves, traveling parallel to the layering in an unbounded medium, is sufficiently well described by this theory. It was pointed out by Herrmann and Achenbach [3] that this approximate theory, which was termed the "effective stiffness theory", bears close resemblance to theories of linear elasticity with micro-structure.

In this paper the simplest possible version of the effective stiffness theory, which was discussed in Ref. [4], is used to describe wave propagation phenomena. It is assumed, however, that the soft layers have random inhomogeneities, i.e. the Lamé parameters and the density of the soft layers fluctuate in a random manner about the constant mean values. Since the spatial derivatives of the elastic parameters do not vanish, additional terms appear in the displacement equations of motion, and the equations have become linear partial differential equations with stochastic coefficients. Their solutions are, therefore, random waves, and the displacement is a random vector-valued function **u** of the position vector and time. In this paper statistical information on the above-mentioned stochastic process is obtained by assuming that the statistical properties of the random inhomogeneities are known.

Keller [5] has developed a perturbation theory for linear stochastic equations which generally leads in an "honest way"§ to deterministic equations for the expected solution. Whenever the stochastic deviations of the random coefficients are small in comparison to their mean values, this perturbation theory yields a mean solution, which is exact within the framework of the correlation theory. In the relatively simple case of harmonic elastic

† Part of this paper was presented under the title "Wave Propagation in Laminated Random Media" at the IUTAM 12th International Congress of Applied Mechanics, Stanford, California, August 26–31, 1968.

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§ This term was created by Keller, *cf.* Ref. [1], p. 228.

wave motion in linear elastic, random inhomogeneous media, Karal and Keller [6] employed this theory to study plane wave motion of the expected displacement vector. As a main result the dispersion equations for free harmonic mean waves were derived, which separate into equations for mean longitudinal and mean transverse wave types, respectively. As a consequence of the inhomogeneity of the material, the mean waves are found to be dispersive and attenuated, i.e. the wave number of the mean wave is a complex number and the real part is a function of the frequency. The perturbation theory for the scalar wave equation with random coefficients is also extensively discussed in [5] and extended by Chen [7] to propagation of scalar waves in discontinuous random media.†

In this paper the above-mentioned perturbation theory is applied to the stochastic displacement equations of motion to derive deterministic, i.e. sure equations for the expected wave or, with a different viewpoint, to define a deterministic but artificial material from the statistical properties of the random material, in which then the mean wave propagates alone. The mean wave solution—or coherent wave—is not a complete solution. The perturbation theory is, however, also applicable to derive higher order wave approximations, e.g. to find approximate stochastic equations for the incoherent random wave, from which the variance of the displacement and various correlation functions can be calculated, thus describing the total solution in a more complete manner. In this paper however, the attention is focused on the properties of the coherent wave portion. Hierarchy equations of statistical moments,‡ illustrating the properties of the stochastic incoherent wave portion in connection with laminated random materials, will be the subject of a forthcoming paper.

The main analytical results of the theory of mean waves will be dispersion relations of plane harmonic mean waves propagating parallel and perpendicular to the layering. These relations will not be established until after making considerable simplifying assumptions, e.g. the inhomogeneity of the matrix material will be described by a single stochastic process with sample functions varying over the layer thickness only. Using the spectral density function and the formulas of Wiener and Khinchine in Section 5 of this paper, the process is restricted to an ergodic process, and later on being specified as a homogeneous band limited "flat spectrum" process. Although other types of stochastic inhomogeneity processes are discussed briefly, the analysis will be worked out in detail only for the above specified process. The choice of a "white noise" inhomogeneity process as a limiting case is not compatible with Keller's perturbation theory, which will be discussed in Section 5 as well.

2. THE DISPLACEMENT EQUATIONS OF MOTION

The version of the effective stiffness theory, which will be used in this paper, was derived in [3], equations (34), by simplifying the general theory, and was constructed in a heuristic manner in [4], for a body with periodically arranged layers of soft and stiff materials. The assumptions of this simpler theory restrict the interaction between reinforcing sheets and matrix layers, by allowing the matrix layers to store strain energy as an elastic continuum, and permitting the reinforcing sheets to store strain energy only from uniaxial deformation and from flexure. By limiting the further analysis to plane strain and

† A comprehensive survey of literature is given in "Wave Propagation in Random Media" by U. Frisch in Ref. [8].

‡ Briefly discussed in context with scalar processes by Keller [5], Section 5.

assuming plane stress throughout the thickness of the reinforcing sheets, the following energy densities for the reinforcing and matrix layers are then obtained:

Strain energy densities:

$$2U_f = \frac{\left[C_f\left(\frac{\partial u}{\partial x}\right)^2 + D_f\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]}{h} \tag{1.1}$$

$$2U_m = \lambda_m\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)^2 + \mu_m\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2\right]. \tag{1.2}$$

Kinetic energy densities:

$$2T_f = \rho_f(\dot{u}^2+\dot{w}^2) \tag{1.3}$$

$$2T_m = \rho_m(\dot{u}^2+\dot{w}^2). \tag{1.4}$$

In equation (1.1) C_f and D_f denote stretching and bending stiffnesses of the reinforcing sheets, respectively, being defined as

$$C_f = \frac{hE_f}{(1-\nu_f^2)}, \qquad D_f = \frac{h^2 C_f}{12} \tag{1.5, 1.6}$$

where h is the constant thickness of the reinforcing sheets, and the constants E_f and ν_f are Young's modulus and Poisson's ratio, respectively, of the stiff reinforcing material. The constant mass density of this material is denoted by ρ_f. The properties of the inhomogeneous soft matrix layers of constant thickness H are defined by Lamé's coefficients $\lambda_m(x, z)$ and $\mu_m(x, z)$ and the material density $\rho_m(x, z)$. The coordinate system (x, z) and the corresponding displacements (u, w) are shown in Fig. 1. From the expressions (1.1) to (1.4) the approximate total strain and kinetic energy density of the laminated composite is calculated by taking the weighted sum of the strain and kinetic energy densities of the two layers:

$$U = \eta U_f + (1-\eta)U_m \tag{1.7}$$

$$T = \eta T_f + (1-\eta)T_m \tag{1.8}$$

where

$$\eta = h/(h+H). \tag{1.9}$$

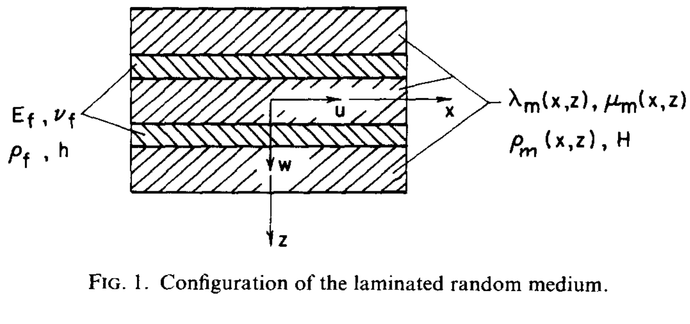

FIG. 1. Configuration of the laminated random medium.

The energies stored in a volume R of the laminated medium are achieved by integrating the densities (1.7, 1.8) over this volume. Finally, by employing Hamilton's principle, see for example Ref. [4], and by taking into account that λ_m, μ_m and ρ_m are not constants, the following stochastic displacement equations of motion are obtained:

$$(v_{m1}^2+v^2)\frac{\partial^2 u}{\partial x^2}+v_{m2}^2\frac{\partial^2 u}{\partial z^2}+(v_{m1}^2-v_{m2}^2)\frac{\partial^2 w}{\partial x\,\partial z} +\alpha^2\left[\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\frac{\partial \mu_m}{\partial z}+\frac{\partial u}{\partial x}\frac{\partial(\lambda_m+2\mu_m)}{\partial x}+\frac{\partial w}{\partial z}\frac{\partial \lambda_m}{\partial x}\right]=\frac{\partial^2 u}{\partial t^2} \tag{1.10}$$

$$(v_{m1}^2-v_{m2}^2)\frac{\partial^2 u}{\partial x\,\partial z}+v_{m2}^2\frac{\partial^2 w}{\partial x^2}+v_{m1}^2\frac{\partial^2 w}{\partial z^2}-l^2v^2\frac{\partial^4 w}{\partial x^4} +\alpha^2\left[\frac{\partial u}{\partial x}\frac{\partial \lambda_m}{\partial z}+\frac{\partial w}{\partial z}\frac{\partial(\lambda_m+2\mu_m)}{\partial z}+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\frac{\partial \mu_m}{\partial x}\right]=\frac{\partial^2 w}{\partial t^2} \tag{1.11}$$

where

$$\rho_c = \eta\rho_f+(1-\eta)\rho_m \tag{1.12}$$

$$\alpha^2 = (1-\eta)/\rho_c \tag{1.13}$$

$$v_{m1}^2 = \alpha^2(\lambda_m+2\mu_m) \tag{1.14}$$

$$v_{m2}^2 = \alpha^2\mu_m \tag{1.15}$$

$$v^2 = \eta C_f/(h\rho_c) \tag{1.16}$$

and

$$l^2 = h^2/12. \tag{1.17}$$

The characteristic length l relates the effective stiffness theory to a simple version of the theory of elasticity with micro-structure.† The mathematical difference between the displacement equations of motion (1.10, 1.11), based on an effective stiffness theory, and the corresponding equations of an effective modulus theory‡ is the appearance of spatial derivatives of order higher than two. In the simple theory presented here a fourth order term is in equation (1.11). It was shown by Achenbach [4], who derived equations (1.10, 1.11) for constant matrix-material properties, that the dispersion of the lowest mode of vertically polarized shear waves, i.e. SV-waves, propagating parallel to the layering, can be approximately described. The dispersion of other wave types with propagation directions either parallel or perpendicular to the layering, which is expected to occur in a laminated composite as well, is, however, beyond the range of this simple theory. The more general effective stiffness theory employed in [2] for homogeneous matrix material describes the dispersion properties of various wave types, propagating either parallel or perpendicular to the layering, thereby improving the theory employed in this paper to a great extent. Further improvement, as pointed out in [2], can only be accomplished by considering the higher wave modes in the approximate displacement distributions. Since this is a first study of the influence of inhomogeneities in the matrix layers, the simplest version of the effective

† *Cf.* [3], particularly equation (35).

‡ *Cf.*, e.g. the references numbered 1 through 4 of [2].

stiffness theory, equations (1.10, 1.11), should be adequate to show the effects on dispersion and attenuation of those waves.

The equations (1.10, 1.11) are linear partial differential equations with stochastic coefficients $\lambda_m(x, z)$, $\mu_m(x, z)$, $\rho_m(x, z)$. To decrease the number of parameters, the inhomogeneity of the soft matrix material is restricted assuming v_m and ρ_m as constants. Hence

$$\alpha^2 = \text{const}, \quad \lambda_m = \bar{v}\mu_m, \qquad \bar{v} = \frac{2v_m}{(1-2v_m)} = \text{const}. \tag{1.18, 1.19}$$

In this manner only the shear modulus is left to be prescribed in an appropriate way. In view of Keller's method employed in the following paragraph, it is assumed that the shear modulus can be expressed as

$$\mu_m(x, z) = \bar{\mu} + \varepsilon\mu(x, z) \tag{1.20}$$

where $\bar{\mu}$ is the constant mean value and $\mu(x, z)$ is the random deviation.

3. KELLER'S PERTURBATION THEORY

In [5], Keller developed a perturbation procedure for linear stochastic operators which generally leads to deterministic equations for the expected solution. This procedure is employed for the displacement equations of motion (1.10, 1.11), which can be written in matrix form as

$$\mathbf{Mu} = 0 \tag{2.1}$$

where $\mathbf{M}$ is a stochastic partial differential matrix operator, which, in view of equation (1.20), can be expressed as the sum of two operators

$$\mathbf{M} = \mathbf{L} + \varepsilon\mathbf{L}_1 \tag{2.2}$$

where $\mathbf{L}$ is the constant coefficient deterministic, i.e. sure operator for the case of homogeneous soft layers with constant material parameters $\bar{\lambda}$, $\bar{\mu}$ and ρ_m, whereas ε is a measure of the departure of the material properties of the soft layers from the homogeneity which is expressed by equation (1.20), and $\mathbf{L}_1$ is a perturbing random operator representing the effects of inhomogeneity. After some manipulations and taking expectations in equation (2.1) by inserting equation (2.2), the following deterministic equation for the mean wave $\langle \mathbf{u} \rangle$ is derived:†

$$(\mathbf{L} - \varepsilon^2 \langle \mathbf{L}_1 \mathbf{L}^{-1} \mathbf{L}_1 \rangle) \langle \mathbf{u} \rangle = 0 + o(\varepsilon^3) \tag{2.3}$$

which becomes an explicit equation for $\langle \mathbf{u} \rangle$ when the $o(\varepsilon^3)$ term is omitted. It may be of interest to note that equation (2.3) does not depend on the sign of ε in equation (2.2). The operator $\mathbf{L}^{-1}$, the inverse of the operator $\mathbf{L}$, appearing in equation (2.3) can be represented as an integral operator. Its kernel is the Green's dyadic $G(\mathbf{r}, \mathbf{r}')$ of the background medium.

† *Cf.* Keller [5], equation (11) with $\langle \mathbf{L}_2 \rangle = 0$.

For future reference the matrix operators $\mathbf{L}$ and $\mathbf{L}_1$, which are obtained from equations (1.10, 1.11), are written explicitly as

$$\mathbf{L} = \begin{Bmatrix} L_u & (\bar{v}_1^2-\bar{v}_2^2)\dfrac{\partial^2}{\partial x\,\partial z} \\ (\bar{v}_1^2-\bar{v}_2^2)\dfrac{\partial^2}{\partial x\,\partial z} & L_w \end{Bmatrix}, \qquad \mathbf{L}_1 = \begin{Bmatrix} L_{u1} & L_{w1} \\ L_{u2} & L_{w2} \end{Bmatrix} \tag{2.4a, b}$$

where the operators with constant coefficients in $\mathbf{L}$ are given by

$$L_u = \bar{a}^2\frac{\partial^2}{\partial x^2}+\bar{v}_2^2\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2} \tag{2.5}$$

$$L_w = \bar{v}_2^2\frac{\partial^2}{\partial x^2}+\bar{v}_1^2\frac{\partial^2}{\partial z^2}-l^2v^2\frac{\partial^4}{\partial x^4}-\frac{\partial^2}{\partial t^2}, \tag{2.6}$$

and the operators with stochastic coefficients in L_1 are expressed by

$$L_{u1} = v_1^2\frac{\partial^2}{\partial x^2}+v_2^2\frac{\partial^2}{\partial z^2}+\alpha^2\left[\frac{\partial\mu}{\partial z}\frac{\partial}{\partial z}+(\bar{v}+2)\frac{\partial\mu}{\partial x}\frac{\partial}{\partial x}\right] \tag{2.7}$$

$$L_{u2} = b^2\frac{\partial^2}{\partial x\,\partial z}+\alpha^2\left[\bar{v}\frac{\partial\mu}{\partial z}\frac{\partial}{\partial x}+\frac{\partial\mu}{\partial x}\frac{\partial}{\partial z}\right] \tag{2.8}$$

$$L_{w1} = b^2\frac{\partial^2}{\partial x\,\partial z}+\alpha^2\left[\frac{\partial\mu}{\partial z}\frac{\partial}{\partial x}+\bar{v}\frac{\partial\mu}{\partial x}\frac{\partial}{\partial z}\right] \tag{2.9}$$

$$L_{w2} = v_2^2\frac{\partial^2}{\partial x^2}+v_1^2\frac{\partial^2}{\partial z^2}+\alpha^2\left[(\bar{v}+2)\frac{\partial\mu}{\partial z}\frac{\partial}{\partial z}+\frac{\partial\mu}{\partial x}\frac{\partial}{\partial x}\right]. \tag{2.10}$$

The constants $\bar{a}^2$, $\bar{b}^2$ and b^2 are

$$\bar{a}^2 = \bar{v}_1^2+v^2, \qquad \bar{b}^2 = \bar{v}_1^2-\bar{v}_2^2 = \alpha^2(\bar{v}+1)\bar{\mu}, \qquad b^2 = v_1^2-v_2^2 = \alpha^2(\bar{v}+1)\mu(x,z) \tag{2.11}$$

and the coefficients $\bar{v}_1^2$, $\bar{v}_2^2$ and v_1^2, v_2^2 are expressed by equations (1.14, 1.15) replacing λ_m, μ_m by the constant values $\bar{\lambda}$, $\bar{\mu}$, and $\lambda(x,z)$, $\mu(x,z)$, respectively, and omitting the index m on the left side. Hence, because of assumption (1.19) it follows, e.g. $v_1^2 = \alpha^2(v+2)\mu(x,z)$, etc.

Equation (2.3) was solved by Karal and Keller [6], Section 4, for much simpler operators $\mathbf{L}$, $\mathbf{L}^{-1}$, $\mathbf{L}_1$, which were appropriate to a randomly inhomogeneous isotropic linear elastic infinite medium, by assuming $\langle\mathbf{u}(\mathbf{r})\rangle$ as a plane wave given by

$$\langle\mathbf{u}(\mathbf{r})\rangle = \mathbf{A}\,e^{i\omega t}\,e^{i\mathbf{k},\mathbf{r}}. \tag{2.12}$$

As a general result the longitudinal and transversal mean waves were found to be uncoupled, dispersive, and attenuated. Special results were obtained by choosing a special form for the autocorrelation function of the homogeneous density process

$$R(r) = \langle\rho^2\rangle\,e^{-r/a} \tag{2.13}$$

and assuming constant elastic parameters. Similar results were derived for electro-magnetic waves, specializing $\mathbf{L}$, $\mathbf{L}^{-1}$, and $\mathbf{L}_1$ from Maxwells' equations. A fast growing literature exists on wave propagation in "weakly" random media,† especially in the fields of electro-magnetic waves (cf. Liu [9]), and above all in scalar wave propagation, where the approach of Chernov [10] is successfully applied. The recent article by Frisch in [8] covers the entire field of wave propagation in random media. Chen and Tien [11] applied equation (2.3) to the problem of heat conduction in a half-space with random thermal diffusivity, specializing $\mathbf{L}$, $\mathbf{L}^{-1}$, and $\mathbf{L}_1$ from the one-dimensional heat conduction equation.

Equation (2.3) must always be completed by the determination of the inverse operator $\mathbf{L}^{-1}$, which is expressed through Green's matrix and depends on the parameters of the background medium. The effective stiffness theory of laminated solids provides a kind of "homogeneous" background medium because the operator $\mathbf{L}$, *cf.* equation (2.6a), has constant coefficients. In the subsequent section the Green's function of a laminated background medium consisting of double layers of homogeneous soft and stiff materials is considered.

4. GREEN'S DYADIC FOR A LAMINATED BACKGROUND MEDIUM WITH HOMOGENEOUS STIFF AND SOFT LAYERS

The Green's dyadic of the background medium is the solution of the inhomogeneous deterministic equation

$$\mathbf{L}^*\mathbf{G}(x, z; x', z') = \mathbf{I}\delta(x-x')\delta(z-z') \tag{3.1}$$

where the time dependence of the source term has been omitted and $\mathbf{L}^*$ is the corresponding time reduced operator $\mathbf{L}$, expressed by equation (2.4a). The identity matrix is denoted by $\mathbf{I}$. Considering only waves of the form (2.12), a corresponding time harmonic source is introduced by equation (3.1). The solution of this equation is easily obtained by employing the double Fourier transformation with respect to x and z. Hence, multiplying the equation (3.1) by $\exp(-i\xi x)$ and $\exp(-i\zeta z)$, integrating then with respect to x and z from minus infinity to infinity, solving further the resulting system of linear algebraic equations for components of the transformed dyadic, we obtain

$$\bar{\bar{\mathbf{G}}}(\xi, \zeta) = \begin{Bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{Bmatrix}, \tag{3.2}$$

wherein

$$G_{11} = -(-\omega^2 + \bar{v}_2^2\xi^2 + \bar{v}_1^2\zeta^2 + l^2v^2\xi^4)/N \tag{3.3}$$

$$G_{12} = \bar{b}^2\xi\zeta/N \tag{3.4}$$

$$G_{22} = (-\omega^2 + \bar{a}^2\xi^2 + \bar{v}_2^2\zeta^2)/N \tag{3.5}$$

$$N = [-\omega^2 + \bar{a}^2\xi^2 + \bar{v}_2^2\zeta^2][-\omega^2 + \bar{v}_2^2\xi^2 + \bar{v}_1^2\zeta^2 + l^2v^2\xi^4] - (\bar{b}^2\xi\zeta)^2. \tag{3.6}$$

The inversion of expression (3.3 through 3.5) is given by the double integrals

$$\bar{\bar{G}}_{ij}(x_1, z_1) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{ij}(\xi, \zeta)\exp[i(\xi x_1 + \zeta z_1)]\,\mathrm{d}\xi\,\mathrm{d}\zeta \qquad (i, j) = 1, 2 \tag{3.7}$$

† The term is taken from papers dealing with electro-magnetic waves. Weakly random media are defined as media having only small random deviations from a homogeneous "background" medium. The background medium has the average material parameters. In [9] even a background medium which has deterministic inhomogeneities is considered.

where

$$x_1 = x - x', \qquad z_1 = z - z'. \tag{3.8}$$

The procedures for evaluating these integrals are not further discussed, because in the course of this paper only the transformed expressions (3.3 through 3.5) are required in establishing equation (2.3).

5. AVERAGE MOTION IN THE LAMINATED RANDOM MEDIUM

In evaluating equation (2.3) the interest is focused on the basic solutions of the mean waves, especially on the time harmonic train of plane waves

$$\langle \mathbf{u} \rangle = \mathbf{A}\, \mathrm{e}^{i\omega t}\, \mathrm{e}^{ik(n_1 x + n_2 z)} \tag{4.1}$$

where n_1, n_2 denote the directional cosines of the propagation direction with respect to the x- and z-axes, and ω and k denote circular frequency and wave number, respectively. The phase velocity is determined by

$$c_P = \omega/k. \tag{4.2}$$

With the basic solution (4.1) in mind, equation (2.3) can be handled in a straightforward way without explicit knowledge of the Green's matrix in the original space of the background medium.†

The average expression $\langle \mathbf{L}_1 \mathbf{L}^{-1} \mathbf{L}_1 \rangle \langle \mathbf{u} \rangle$ in equation (2.3) can be evaluated in a simpler way considering the identity

$$\langle \mathbf{L}_1 \mathbf{L}^{-1} \mathbf{L}_1 \rangle \langle \mathbf{u} \rangle \equiv \langle \mathbf{L}_1 \mathbf{L}^{-1} \mathbf{L}_1 \langle \mathbf{u} \rangle \rangle \tag{4.3}$$

and equation (4.1). Inserting the Green's dyadic with components (3.7) into (4.3), it renders the expressions

$$\langle \mathbf{L}_1 \mathbf{L}^{-1} \mathbf{L}_1 \rangle \langle \mathbf{u} \rangle \equiv \mathrm{e}^{ik(n_1 x + n_2 z)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-ik(n_1 x_1 + n_2 z_1)} \begin{Bmatrix} F_1(x_1, z_1) \\ F_2(x_1, z_1) \end{Bmatrix} \mathrm{d}x_1\, \mathrm{d}z_1 \tag{4.4}$$

where F_1 and F_2 depend on the various correlation functions of the random processes $\mu(x, z)$ and $\mu'(x, z)$ and‡ on the first and second orders spatial derivatives of the Green's functions $\bar{\bar{G}}_{ij}(x_i, z_i)$. In (4.4) the two-dimensional random process $\mu(x, z)$ is assumed to be defined by the product

$$\mu(x, z) = \mu_1(x)\mu_2(z) \tag{4.5}$$

where μ_1 and μ_2 are statistically independent homogeneous random processes. To simplify the further analysis with respect to the number of statistical parameters involved and because of the greater physical significance only the z-dependence is taken into consideration. Hence,

$$\mu(x, z) \equiv \mu(z). \tag{4.6}$$

† The subsequently demonstrated procedure simplifies also the problems treated in [6], where the Green's dyadic in its natural form is considered.

‡ The prime denotes the partial derivative with respect to x and z, respectively.

For this case the functions F_1 and F_2 are listed in Appendix A.† The autocorrelation function of the homogeneous process (4.6) depends solely on $z_1 = z - z'$ and is denoted by ‡

$$\langle \mu(z)\mu(z')\rangle = R(z_1). \tag{4.7}$$

Furthermore,

$$\langle \mu'(z)\mu'(z')\rangle = S(z_1) \tag{4.8}$$

denotes the covariance of the process $d\mu/dz$, and

$$\langle \mu(z)\mu'(z')\rangle = T(z_1), \qquad \langle \mu'(z)\mu(z')\rangle = \bar{T}(z_1) \equiv T(-z_1) \tag{4.9}$$

denote the cross correlation functions of the processes $d\mu/dz$ and μ, respectively. The application of the operator **L**, equation (2.4a), to the expression (4.1), and taking into account the partial result (4.4), yields the coupled linear equations

$$\begin{aligned} &A[\omega^2 + (ik)^2(\bar{a}^2 n_1^2 + \bar{v}_2^2 n_2^2)] + B(ik)^2 \bar{b}^2 n_1 n_2 \\ &\quad = \varepsilon^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x_1, z_1)\, e^{-ik(n_1 x_1 + n_2 z_1)}\, dx_1\, dz_1 \\ &A(ik)^2 \bar{b}^2 n_1 n_2 + B[\omega^2 + (ik)^2(\bar{v}_2^2 n_1^2 + \bar{v}_1^2 n_2^2) - l^2 v^2 (ikn_1)^4] \\ &\quad = \varepsilon_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x_1, z_1)\, e^{-ik(n_1 x_1 + n_2 z_1)}\, dx_1\, dz_1 \end{aligned} \tag{4.10}$$

where the factor $\exp[i(\omega t + kn_1 x + kn_2 z)]$ has been cancelled. The kernel functions $F_1(x_1, z_1)$ and $F_2(x_1, z_1)$ are listed in Appendix A.

The left hand sides of these equations are exactly the equations derived by Achenbach in [4]. It can easily be verified from equation (4.10) with $\varepsilon = 0$, that various wave modes can be treated independently, e.g. choosing $B = 0, n_2 = 0, n_1 = 1$ and $A = 0, n_1 = 0, n_2 = 1$ leads to the solution of P-waves traveling parallel and perpendicular to the layering, the special values $A = 0$, $n_2 = 0$, $n_1 = 1$ and $B = 0$, $n_1 = 0$, $n_2 = 1$ determine S-waves travelling parallel as SV- waves and perpendicular to the layering. From these wave modes only the SV-type is found to be dispersive, which is a consequence of the simple version of the effective stiffness theory employed. From equation (4.10) the SV-dispersion relation for the background medium reads because of $B \neq 0$ like this

$$\omega^2 - (k\bar{v}_2)^2 - (lvk^2)^2 = 0 \tag{4.11}$$

or in terms of the phase velocity (4.2)

$$c_p^2 = \bar{v}_2^2 + (vk_1)^2 \tag{4.12}$$

where $k_1 = kl$ is a dimensionless wave number, l expressed by equation (1.17). To improve the results of equation (4.12), $\bar{v}_2^2$ is replaced by the known squared value of the limiting

† It should be noted that the subsequently demonstrated procedure for the further evaluation of the expression (4.4) is equally well applicable to the more general case when equation (4.5) is valid.

‡ For details *cf.* Ref. [12], chapter 18.

phase velocity for vanishing wave number of SV-waves according to the exact solution as derived in [15], equation (17), as follows

$$\bar{v}_2^2 \to \bar{v}_{21}^2 = \frac{\bar{\mu}}{[\rho_c(1-\eta+\eta\bar{\mu}/\mu_f)]} \tag{4.13}$$

where μ_f is the shear modulus of the stiff reinforcing sheets. From equations (1.15) and (1.12, 1.13) it follows with $\bar{\mu}$ instead of μ_m that

$$\bar{v}_2^2 = \frac{(1-\eta)\bar{\mu}}{\rho_c} \tag{4.14}$$

which is independent of the elastic material parameters of the stiff layers. An improvement of the present theory can be achieved by replacing all the parameters $\bar{a}^2$, $\bar{v}_1^2$, and the above discussed $\bar{v}_2^2$ by the corresponding limiting phase velocities for vanishing wave numbers of the special single wave systems with propagation directions parallel or perpendicular to the layering, respectively, as tabulated, e.g. in [2].

The Fourier type integrals, which enter the right hand side of equations (4.10), represent the effect of the random inhomogeneities of the soft material layers up to the order ε^2. A close inspection of these integrals, $F_{1,2}$ given in Appendix A, reveals that they can be reduced to a single integration of the functions G_{ij}, equations (3.3 through 3.5), weighted by a factor which can be expressed by the known spectral density functions of the processes $\mu(z)$ and $\mu'(z)$. The spectral density of, an, at least, weakly stationary, i.e. homogeneous, and ergodic random process is related to its autocorrelation function by the formulas of Wiener and Khinchine, see, e.g. Lin [14], p. 58:†

$$\Phi(s) = \int_{-\infty}^{\infty} R(z_1)\,e^{isz_1}\,dz_1, \qquad R(z_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(s)\,e^{-isz_1}\,ds \tag{4.15}$$

where the spectral density function is denoted by $\phi(s)$. Hence, as an additional assumption the process $\mu(z)$ is understood as an ergodic process, see, e.g. [13]. The spectral densities of $\mu(z)$ and $\mu'(z)$ will be denoted by $\phi(s)$ and $\psi(s)$, respectively, and the cross correlation of these two processes will be set zero in the further analysis.‡ For the sake of brevity the procedure of simplifying the integrals $F_{1,2}$ is only shown for the first two non-zero terms of $F_{1,2}$ which are characteristic of all the terms forming $F_{1,2}$; see Appendix A. Denoting the integrals by

$$\mathrm{I} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x_1, z_1)\,e^{-ik(n_1x_1+n_2z_1)}\,dx_1\,dz_1 = \sum_{i=1}^{10} \mathrm{I}_i \tag{4.16}$$

$$\mathrm{II} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x_1, z_1)\,e^{-ik(n_1x_1+n_2z_1)}\,dx_1\,dz_1 = \sum_{i=1}^{10} \mathrm{II}_i \tag{4.17}$$

the first term of F_1 integrated with respect to x_1 renders

$$\begin{aligned}\mathrm{I}_1 &= a_{11}\alpha^4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2\bar{\bar{G}}_{11}(x_1,z_1)}{\partial x_1^2}R(z_1)\,e^{-i(\xi x_1+\zeta z_1)}\,dx_1\,dz_1\\ &= a_{11}\alpha^4(i\xi)^2\int_{-\infty}^{\infty}\bar{G}_{11}(\xi, z_1)R(z_1)\,e^{-i\zeta z_1}\,dz_1\end{aligned}$$

†The factor $(2\pi)^{-1}$, for the sake of convenience, is shifted to the inverse integral of $\phi(s)$.

‡ The effect of correlation of the processes $\mu(z)$ and $\mu'(z)$ will be discussed after equation (4.26)

where

$$\xi = kn_1, \qquad \zeta = kn_2. \tag{4.18}$$

Then, using equation (4.15) the second transformation can be performed as follows

$$\begin{aligned} \mathrm{I}_1 &= a_{11}\alpha^4(i\xi)^2 \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{G}_{11}(\xi, z_1)\Phi(s)\,\mathrm{e}^{-i(s+\zeta)z_1}\,\mathrm{d}z_1\,\mathrm{d}s \\ &= a_{11}\alpha^4(i\xi)^2\frac{1}{2\pi}\int_{-\infty}^{\infty} G_{11}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s \end{aligned} \tag{4.19}$$

where G_{11} is expressed by (3.3) if ζ is replaced by $\zeta + s$. Although the second term of F_1 includes derivatives of $\bar{\bar{G}}_{11}$ with respect to z_1, it can be handled in a similar way as demonstrated above. The interchange of integrals renders in this case

$$\begin{aligned} \mathrm{I}_2 &= b_{11}\alpha^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{s^2\bar{\bar{G}}_{11}(x_1, z_1)}{\partial z_1^2} R(z_1)\,\mathrm{e}^{-i(\xi x_1 + \zeta z_1)}\,\mathrm{d}x_1\,\mathrm{d}z_1 \\ &= b_{11}\alpha^4\frac{1}{2\pi}\int_{-\infty}^{\infty} [i(s+\zeta)]^2 G_{11}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s \end{aligned} \tag{4.20}$$

where a factor of highest order s^2 appears in the integrand. The expressions I_3 through I_{10} and II_1 through II_{10} evaluated in a similar manner are listed in Appendix B. A necessary condition for the existence of the various integrals over an infinite range of s is the vanishing of the integrands at infinity. This condition excludes the choice of a white-noise process $\mu(z)$ since $\phi(s)$ = const for all s. The autocorrelation function is given by

$$R(z_1) = \delta(z_1). \tag{4.21}$$

The more realistic case of $\mu(z)$ being a stationary band limited "flat spectrum" process will be studied in the further analysis. It has the properties

$$\Phi(s) = \begin{cases} \Phi_0 & |s| \le a \\ 0 & |s| > a \end{cases} \tag{4.22}$$

$$R(z_1) = \frac{a\Phi_0}{\pi}\frac{\sin(az_1)}{az_1} \tag{4.23}$$

and renders a finite range of s-integration of the integrals I, II. The exponential function (2.13) denotes the autocorrelation function of a so-called "random telegraph wave" process.† The corresponding spectral density has the form

$$\Phi(s) = \frac{2aR(0)}{1+(as)^2}, \qquad -\infty < s < \infty$$

where a is a measure of the correlation distance of the $\mu(z)$-process. For $s \to \infty$, $\Phi(s) \sim s^{-2}$, and the corresponding integrands in I and II converge to zero for $s \to \infty$. A table of other frequently applied correlation functions with their corresponding spectral densities is presented as Appendix III in Lin [14].

† The choice of this process implies an additional layering within the matrix layers, where $\mu(z)$ takes on the values $\pm\sqrt{R(0)}$ over layer thicknesses which are randomly distributed. The sign changes are generated by the state changes of a Poisson process of mean count rate $1/2a$, i.e. the probability that no sign changes take place in distance $Z > 0$ is given by $\exp(-Z/2a)$.

Further evaluations of the integrals I_i and II_i require a specialization of $\phi(s)$, which will be chosen, for the sake of convenience, in the form (4.22). Hence, the process $\mu(z)$ can be represented by the series expansion

$$\mu(z) = \sum_{k=-\infty}^{\infty} \mu_k \frac{\sin a(z-z_k)}{a(z-z_k)}, \qquad \mu_k \equiv \mu(z_k), z_k = k\pi/a, k = 0, \pm 1, \pm 2, \ldots$$

which converges in the mean-square sense with the sample values μ_k having zero mean and being uncorrelated.† After inserting equations (3.3), (4.22) into (4.19), I_1 becomes

$$\mathrm{I}_1 = -a_{11}\alpha^4\xi^2(\Phi_0/2\pi)\int_{-a}^{a} G_{11}(\xi, s+\zeta)\,\mathrm{d}s \tag{4.24}$$

and (4.20) renders

$$\mathrm{I}_2 = -b_{11}\alpha^4(\Phi_0/2\pi)\int_{-a}^{a} (s+\zeta)^2 G_{11}(\xi, s+\zeta)\,\mathrm{d}s. \tag{4.25}$$

Similar expressions are derived for all the other integrals I_i and II_i being dependent on basic integrals of the form

$$\mathrm{III}_{k,ij} = \int_{-a}^{a} s^k G_{ij}(\xi, s+\zeta)\,\mathrm{d}s, \qquad k = 0, 1, 2, 3 \tag{4.26}$$

which after evaluation render transcendental functions of ξ and ζ and the probabilistic parameter a. This method can be applied in the same manner to the integrals arising from non-zero cross-correlation functions $T(z_1)$ and $\bar{T}(z_1)$ with the associated cross spectral densities $\pm is\Phi(s)$. The non-zero contributions to I and II are real and thus the waves remain unattenuated. Hence, the consideration of the correlated processes $\mu(z)$ and $\mu'(z)$ will not influence the character of the results derived later.

A. *Dispersion relations of plane waves propagating perpendicular to the layering*

Corresponding to the uncoupled plane wave systems travelling parallel or perpendicular to the layering in the background medium, the uncoupled plane mean waves propagating perpendicular to the layering of the averaged stochastic medium will be treated first. They become dispersive from the random inhomogeneities of the matrix material, yet remain unattenuated. For these waves $n_1 = 0$ and $n_2 = 1$. Since $\xi = 0$ and $\zeta = k$, equations (4.10) yield

$$A\left\{\omega^2 - (k\bar{v}_2)^2 + \varepsilon^2 \frac{k\alpha^4}{2\pi}\Phi_0 \int_{-a}^{a} \frac{k(s+k)^2 - s^2(s+k)}{\bar{v}_2^2(s+k)^2 - \omega^2}\,\mathrm{d}s\right\} = 0 \tag{4.27}$$

$$B\left\{\omega^2 - (k\bar{v}_1)^2 - (2+\bar{\nu})\varepsilon^2 \frac{k\alpha^4}{2\pi}\Phi_0 \int_{-a}^{a} \frac{(2+\bar{\nu})k(s+k)^2 - s^2(s+k)}{\bar{v}_1^2(s+k)^2 - \omega^2}\,\mathrm{d}s\right\} = 0 \tag{4.28}$$

† The properties of the sampling functions are discussed in [12], p. 18.11.

where A is the amplitude of S-waves and B is the amplitude of P-waves. The formal relation $\Psi(s) = s^2\Phi(s)$ is considered in the above given equations. The evaluation of the integrals renders the dispersion relations

$A \neq 0$:

$$\beta_s^2 = \beta_s^2(k \to 0) + \varepsilon^2 \frac{a\Phi_0}{\pi}\left(\frac{\alpha}{\bar{v}_2}\right)^4 \frac{k}{4a}\left[3\beta_s \ln \frac{a(a+2k)-k^2(1-\beta_s^2)}{(-1)^m a(a-2k)-k^2(1-\beta_s^2)} + (1+\beta_s^2)\ln\frac{a(a+2k)+k^2(1-\beta_s^2)}{(-1)^m a(a-2k)+k^2(1-\beta_s^2)}\right] \tag{4.29}$$

$B \neq 0$:

$$\beta_P^2 = \beta_P^2(k \to 0) - \varepsilon^2 \frac{a\Phi_0}{\pi}\left(\frac{\alpha}{\bar{v}_1}\right)^4 \frac{(2+\bar{v})k}{4a}\left[(4+\bar{v})\beta_P \ln \frac{a(a+2k)-k^2(1-\beta_P^2)}{(-1)^m a(a-2k)-k^2(1-\beta_P^2)} + (1+\beta_P^2)\ln\frac{a(a+2k)+k^2(1-\beta_P^2)}{(-1)^m a(a-2k)+k^2(1-\beta_P^2)}\right] \tag{4.30}$$

where m must be chosen even for $0 \leq k < a/2$ and odd for $k > a/2$. The wave number k and the dimensionless phase velocities

$$\begin{aligned} \beta_S &= c_S/\bar{v}_2 \equiv \omega/(k\bar{v}_2) \\ \beta_P &= c_P/\bar{v}_1 \equiv \omega/(k\bar{v}_1) \end{aligned} \tag{4.31}$$

of S- and P-waves, respectively, are real. The limiting phase velocities for $k \to 0$ are given by

$$\beta_S^2(k \to 0) = 1 - 2\varepsilon^2 \frac{a\Phi_0}{\pi}\left(\frac{\alpha}{\bar{v}_2}\right)^4 \tag{4.32}$$

$$\beta_P^2(k \to 0) = 1 + (3+\bar{v})(2+\bar{v})\varepsilon^2 \frac{a\Phi_0}{\pi}\left(\frac{\alpha}{\bar{v}_1}\right)^4. \tag{4.33}$$

The expressions (4.29, 4.30) become simpler when only terms up to the order ε^2 are retained. With $\beta = 1$ within the brackets of these expressions the phase velocities are given explicitly by

$$\beta_S^2 = \beta_S^2(k \to 0) + \varepsilon^2 \frac{a\Phi_0}{\pi}\left(\frac{\alpha}{\bar{v}_2}\right)^4 \frac{5k}{4a} \ln \frac{a+2k}{(-1)^m(a-2k)} \tag{4.34}$$

$$\beta_P^2 = \beta_P^2(k \to 0) - \varepsilon^2 \frac{a\Phi_0}{\pi}\left(\frac{\alpha}{\bar{v}_1}\right)^4 \frac{(2+\bar{v})(6+\bar{v})k}{4a} \ln \frac{a+2k}{(-1)^m(a-2k)} \tag{4.35}$$

where m is chosen even for $0 \leq k < a/2$ and odd for $k > a/2$. Since the dispersion within the background medium is neglected, the results are restricted to long wavelengths solution, i.e. $k(h+H) \ll 1$. In addition, in the case $k \ll a$ the logarithmic functions in those expressions with m even can be approximated by the first term of a Taylor expansion

$$\ln \frac{a+2k}{a-2k} \approx 4k/a + o(k^3/a^3)$$

thus the squared phase velocities become proportional to $(k/a)^2$.

B. *Dispersion relations of plane waves propagating parallel to the layering*

Another more significant case deals with the propagation of mean waves parallel to the layering, which implies $n_1 = 1, n_2 = 0, \xi = k, \zeta = 0$. The evaluation of the integrals (4.26) is much more laborous in this case than in the above treated case, because $\xi = k$ and $\zeta \to s$ inserted in equations (3.3 through 3.6) do not simplify the expressions G_{ij}, which form the integrands

$$G_{11}(k, \omega, s) = -(-\omega^2 + \bar{v}_2^2 k^2 + l^2 v^2 k^4 + \bar{v}_1^2 s^2)/N \tag{4.36}$$

$$G_{12}(k, \omega, s) = \bar{b}^2 ks/N \tag{4.37}$$

$$G_{22}(k, \omega, s) = (-\omega^2 + \bar{a}^2 k^2 + \bar{v}_2^2 s^2)/N \tag{4.38}$$

$$N = [-\omega^2 + \bar{a}^2 k^2 + \bar{v}_2^2 s^2][-\omega^2 + \bar{v}_2^2 k^2 + l^2 v^2 k^4 + \bar{v}_1^2 s^2] - (\bar{b}^2 ks)^2. \tag{4.39}$$

Considering the coefficients in $F_{1,2}$ given in Appendix A, which reduce to the expressions

$$b_{11} = -(2+\bar{v})Ak^2, \qquad b_{21} = ikB \tag{4.40}$$

$$B_{11} = -Bk^2, \qquad B_{21} = ik\bar{v}A \tag{4.41}$$

$$a_{11} = -(2+\bar{v})^2 Ak^2, \qquad A_{11} = -(2+\bar{v})Bk^2 \tag{4.42}$$

$$a_{21} = ik(2+\bar{v})B, \qquad A_{21} = ik\bar{v}(2+\bar{v})A \tag{4.43}$$

equations (4.10) will not immediately decouple to single equations for each of the amplitudes of the P- and SV-waves considered. Terms of order ε^2 lead to coupling between these two waves, and equations (4.10) define the amplitude ratio for each pair of frequency and wave number, for which such a wave system exists in the mean. Equations (4.10) become the homogeneous system

$$A[D_P^2 - \varepsilon^2 m_1(k, \omega, a)] - \varepsilon^2 Bm_2(k, \omega, a) = 0 \tag{4.44}$$

$$-\varepsilon^2 Am_3(k, \omega, a) + [D_S^2 - \varepsilon^2 m_4(k, \omega, a)]B = 0 \tag{4.45}$$

where

$$Am_1(k, \omega, a) + Bm_2(k, \omega, a) = \mathrm{I} \tag{4.46}$$

$$Am_3(k, \omega, a) + Bm_4(k, \omega, a) = \mathrm{II} \tag{4.47}$$

and

$$D_P^2 = \omega^2 - (\bar{a}k)^2, \qquad D_S^2 = \omega^2 - (\bar{v}_2 k)^2 - l^2 v^2 k^4. \tag{4.48}$$

In terms of I_i and II_i, Appendix B, the $m_i(k, \omega, a)$ are given by

$$Am_1(k, \omega, a) = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_4 + \mathrm{I}_8 + \mathrm{I}_{10} \tag{4.49}$$

$$Bm_2(k, \omega, a) = \mathrm{I}_3 + \mathrm{I}_5 + \mathrm{I}_6 + \mathrm{I}_7 + \mathrm{I}_9 \tag{4.50}$$

$$Am_3(k, \omega, a) = \mathrm{II}_1 + \mathrm{II}_4 + \mathrm{II}_5 + \mathrm{II}_6 + \mathrm{II}_{10} \tag{4.51}$$

$$Bm_4(k, \omega, a) = \mathrm{II}_2 + \mathrm{II}_3 + \mathrm{II}_7 + \mathrm{II}_8 + \mathrm{II}_9. \tag{4.52}$$

Although the coefficients $b_{21}, B_{21}, a_{21}, A_{21}$ are complex, the expressions m_1 through m_4 turn out to be real, and the amplitudes A and B retain, therefore, real constants. To evaluate the basic integrals (4.26), the four zeros of the denominator (4.39) are determined through the plus and minus roots of

$$s_{1,2}^2 = \frac{\frac{1}{2}[(\bar{v}_1^2 D_P^2 + \bar{v}_2^2 D_S^2 + (\bar{b}^2 k)^2 \pm S]}{(\bar{v}_1 \bar{v}_2)^2} \tag{4.53}$$

where

$$\begin{aligned} S &\equiv (s_1^2 - s_2^2)(\bar{v}_1 \bar{v}_2)^2 \\ &= \{[\bar{v}_1^2 D_P^2 - \bar{v}_2^2 D_S^2]^2 + (\bar{b}^2 k)^4 + 2(\bar{b}k)^2 [\bar{v}_1^2 D_P^2 + \bar{v}_2^2 D_S^2]\}^{1/2} \end{aligned} \tag{4.54}$$

and$s_2^2 < s_1^2$.

Since $N = (\bar{v}_1 \bar{v}_2)^2 (s^2 - s_1^2)(s^2 - s_2^2)$, equation (4.26) becomes

$$\mathrm{IV}_0 = \int_{-a}^{a} \frac{\mathrm{d}s}{N} = S^{-1}\left[\frac{1}{s_2} \ln S_2 - \frac{1}{s_1} \ln S_1\right] \tag{4.55}$$

$$\mathrm{IV}_2 = \int_{-a}^{a} s^2 \frac{\mathrm{d}s}{N} = S^{-1}[s_2 \ln S_2 - s_1 \ln S_1] \tag{4.56}$$

$$\mathrm{IV}_4 = \int_{-a}^{a} s^4 \frac{\mathrm{d}s}{N} = \frac{2a}{(\bar{v}_1 \bar{v}_2)^2} + S^{-1}[s_2^3 \ln S_2 - s_1^3 \ln S_1] \tag{4.57}$$

where $s_1 > 0$, $s_2 > 0$ and

$$S_1 = \frac{s_1 + a}{(-1)^m (s_1 - a)}, \qquad S_2 = \frac{s_2 + a}{(-1)^n (s_2 - a)}. \tag{4.58}$$

The integers m and n are determined by

$$\left.\begin{array}{ll} a < s_2 < s_1 \ldots & m, n \text{ even} \\ s_2 < a < s_1 \ldots & m \text{ even}, n \text{ odd} \\ s_2 < s_1 < a \ldots & m, n \text{ odd} \end{array}\right\}. \tag{4.59}$$

The integrals (4.26) which contain odd powers of s vanish identically. Considering this fact in the integrals I_i, II_i, it turns out that $m_2(k, \omega, a) \equiv m_3(k, \omega, a) \equiv 0$. The vanishing of these coefficients decouples the equations (4.44, 4.45) into the two independent equations

$$A[D_P^2 - \varepsilon^2 m_1(k, \omega, a)] = 0 \tag{4.60}$$

$$B[D_S^2 - \varepsilon^2 m_4(k, \omega, a)] = 0 \tag{4.61}$$

where the independent dispersion relations for mean P- and mean SV-waves are derived by simply considering $A \neq 0$ and $B \neq 0$, respectively. The coupling between the two random wave modes, which is expected to occur as an effect of the random inhomogeneities, does not affect the corresponding mean waves up to the order ε^2. The evaluated non-zero integrals $\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_4, \mathrm{I}_8, \mathrm{I}_{10}$ and $\mathrm{II}_2, \mathrm{II}_3, \mathrm{II}_7, \mathrm{II}_8, \mathrm{II}_9$ are listed in Appendix C, from which the expressions (4.49) and (4.52) are calculated, rendering

$$m_1(k, \omega, a) = -2(1+\bar{v})\left(\frac{\alpha}{\bar{v}_2}\right)^2 \frac{a\Phi_0}{\pi} (\alpha k)^2 - \frac{a\Phi_0}{\pi} \frac{\alpha^4 k^2}{2aS}\left(\frac{S_{11}}{s_1} \ln S_1 - \frac{S_{12}}{s_2} \ln S_2\right) \tag{4.62}$$

$$m_4(k, \omega, a) = \frac{a\Phi_0}{\pi} \frac{\alpha^4 k^2}{2aS}\left(\frac{S_{41}}{s_1} \ln S_1 - \frac{S_{42}}{s_2} \ln S_2\right) \tag{4.63}$$

where

$$S_{11} = -2(1+\bar{v})\bar{v}_1^2 s_1^4 + [(2+\bar{v})(D_S^2 - k^2\bar{v}_1^2) + \bar{v}D_P^2 + (1+\bar{v})^2(\bar{v}_1 k)^2]s_1^2 + (2+\bar{v})^2 D_S^2 k^2 \quad (4.64)$$

$$S_{41} = [\bar{v}D_S^2 + (2+\bar{v})D_P^2 - (1+\bar{v})^2\bar{v}_2^2 k^2 - (\bar{v}_2 k)^2]s_1^2 + k^2 D_P^2 \quad (4.65)$$

and S_{12}, S_{42} are given by (4.64, 4.65) replacing s_1 by s_2, while D_P^2, D_S^2 are expressed by (4.48). Some interesting properties of m_1 and m_4 can be shown by expanding the logarithmic terms having arguments expressed by (4.58), for $a \ll s_2$ and $s_2 \ll a$, respectively. The first expansion

$$\ln\frac{s_2+a}{s_2-a} = \frac{2a}{s_2}\left[1+\frac{a^2}{3s_2^2}+o\left(\frac{a^4}{s_2^4}\right)\right], \qquad a \ll s_2 \quad (4.66)$$

renders for $k \to 0$ from equations (4.60, 4.61)

$$\beta_P^2(k\to 0) = 1-\varepsilon^2\frac{(2+\bar{v})^2}{2}\left(\frac{\alpha}{\bar{a}}\right)^4\frac{a\Phi_0}{\pi} \quad (4.67)$$

$$\beta_S^2(k\to 0) = 1+\frac{\varepsilon^2}{2}\left(\frac{\alpha}{\bar{v}_2}\right)^4\frac{a\Phi_0}{\pi}+\left[1-\frac{\varepsilon^2}{2}\left(\frac{\alpha}{\bar{v}_2}\right)^4\frac{a\Phi_0}{\pi}\right]l^2\left(\frac{v}{\bar{v}_2}\right)^2 k^2 \quad (4.68)$$

where $\beta_P = c_P/\bar{a}$ and $\beta_S = c_S/\bar{v}_2$ are the dimensionless phase velocities of P- and SV-waves, respectively. A second expansion

$$\ln\frac{s_2+a}{a-s_2} = \frac{2s_2}{a}\left[1+\frac{s_2^2}{3a^2}+o\left(\frac{s_2^4}{a^4}\right)\right], \qquad s_2 \ll a \quad (4.69)$$

renders instead of equations (4.62, 4.63) the expressions

$$m_1(k,\omega,a) = -2(1+\bar{v})\left(\frac{\alpha}{\bar{v}_2}\right)^2\frac{a\Phi_0}{\pi}(\alpha k)^2 - \frac{a\Phi_0}{\pi}\frac{\alpha^4 k^2}{2a(\bar{v}_1\bar{v}_2)^2(s_1^2-s_2^2)}\left(\frac{S_{11}}{s_1}\ln S_1 - \frac{2S_{12}}{a}\right) \quad (4.70)$$

$$m_4(k,\omega,a) = \frac{a\Phi_0}{\pi}\frac{\alpha^4 k^2}{2a(\bar{v}_1\bar{v}_2)^2(s_2^2-s_1^2)}\left(\frac{S_{41}}{s_1}\ln S_1 - \frac{2S_{42}}{a}\right). \quad (4.71)$$

Taking into account terms of the order ε^2 in $\varepsilon^2 m_{1,4}$, these expressions become considerably simpler because D_P^2 and D_S^2 become zero in m_1 and m_4, respectively. From equation (4.53) it follows that

$$(\bar{v}_1\bar{v}_2)^2 s_1^2 = \bar{v}_2^2 D_S^2 + (\bar{b}^2 k)^2 \ldots \text{ is valid in the expression } m_1,$$

$$(\bar{v}_1\bar{v}_2)^2 s_1^2 = \bar{v}_1^2 D_P^2 + (\bar{b}^2 k)^2 \ldots \text{ is valid in the expression } m_4.$$

In either case $s_2 = 0$. Equations (4.64, 4.65) then reduce to

$$\left.\begin{aligned} S_{11} &= -k^4[(1+\bar{v})\bar{v}_1^2 + v^2(1-l^2k^2)]\left[1-\bar{v}\left(\frac{v}{\bar{v}_1}\right)^2\left(1+\frac{4+3\bar{v}}{\bar{v}}l^2k^2\right)\right] \\ S_{12} &= (2+\bar{v})^2 k^4[(1+\bar{v})\bar{v}_1^2 + v^2(1-l^2k^2)] \end{aligned}\right\} \quad (4.72)$$

and

$$\left.\begin{aligned} S_{41} &= \frac{k^4\bar{v}_2^2}{2+\bar{v}}\left[2(1+\bar{v})^3 + (7+14\bar{v}+9\bar{v}^2+2\bar{v}^3)\left(\frac{v}{\bar{v}_2}\right)^2(1-l^2k^2)\right. \\ &\quad \left. + (2+\bar{v})\left(\frac{v}{\bar{v}_2}\right)^4(1-l^2k^2)^2\right] \\ S_{42} &= -k^4\bar{v}_2^2\left[1+\bar{v}+\left(\frac{v}{\bar{v}_2}\right)^2(1-l^2k^2)\right]. \end{aligned}\right\} \quad (4.73)$$

Finally, equations (4.60, 4.61) take on the form

$$\beta_P^2 = (c_P/\bar{a})^2 = 1+\varepsilon^2 \frac{m_1(k,a)}{(\bar{a}k)^2} \tag{4.74}$$

$$\beta_S^2 = (c_S/\bar{v}_2)^2 = 1+\left(\frac{v}{\bar{v}_2}\right)^2 (kl)^2+\varepsilon^2 \frac{m_4(k,a)}{(\bar{v}_2 k)^2} \tag{4.75}$$

where

$$\frac{m_1(k,a)}{(\bar{a}k)^2} = -2(1+\bar{v})\frac{\alpha^4}{(\bar{a}\bar{v}_2)^2}\frac{a\Phi_0}{\pi}+\frac{\alpha^4}{(\bar{a}\bar{v}_2)^2}\frac{a\Phi_0}{\pi}\left(\frac{k}{a}\right)^2$$
$$\times\left\{\frac{a}{2s_1}\left[1-\bar{v}\left(\frac{v}{\bar{v}_1}\right)^2\left(1+\frac{4+3\bar{v}}{\bar{v}}(kl)^2\right)\right]\ln S_1+(2+\bar{v})^2\right\} \tag{4.76}$$

and

$$\frac{m_4(k,a)}{(\bar{v}_2 k)^2} = \frac{\alpha^4}{(\bar{v}_1\bar{v}_2)^2}\frac{a\Phi_0}{\pi}\left(\frac{k}{a}\right)^2\left[1+\bar{v}+(2+\bar{v})\left(\frac{v}{\bar{v}_2}\right)^2(1-k^2l^2)\right]^{-1}\left\{\frac{a}{2s_1}\left[2(1+\bar{v})^3\right.\right.$$
$$+(7+14\bar{v}+9\bar{v}^2+2\bar{v}^3)\left(\frac{v}{\bar{v}_2}\right)^2(1-k^2l^2)+(2+\bar{v})\left(\frac{v}{\bar{v}_2}\right)^4(1-k^2l^2)^2\bigg]\ln S_1$$
$$\left.+(2+\bar{v})\left[1+\bar{v}+\left(\frac{v}{\bar{v}_2}\right)^2(1-k^2l^2)\right]\right\}. \tag{4.77}$$

The limiting velocities for $k \to 0$ in the particular case, where $a \gg s_1 > s_2$ are given by

$$\beta_P^2(k\to 0) = 1-2(1+\bar{v})\varepsilon^2\frac{\alpha^4}{(\bar{a}\bar{v}_2)^2}\frac{a\Phi_0}{\pi} \tag{4.78}$$

$$\beta_S^2(k\to 0) = 1+l^2\left(\frac{v}{\bar{v}_2}\right)^2 k^2+\varepsilon^2\frac{\alpha^4}{(\bar{v}_1\bar{v}_2)^2}\frac{a\Phi_0}{\pi}\left(\frac{k}{a}\right)^2 S_{44} \tag{4.79}$$

where the constant S_{44} depends on the elastic parameters of the background medium, and is given by

$$S_{44} = \left[1+\bar{v}+(2+\bar{v})\left(\frac{v}{\bar{v}_2}\right)^2\right]^{-1}\left[4+9\bar{v}+7\bar{v}^2+2\bar{v}^3\right.$$
$$\left.+(9+15\bar{v}+9\bar{v}^2+2\bar{v}^3)\left(\frac{v}{\bar{v}_2}\right)^2+(2+\bar{v})\left(\frac{v}{\bar{v}_2}\right)^4\right]. \tag{4.80}$$

In equations (4.76, 4.77) the approximation $(a/(2s_1))\ln S_1 = 1$ has been applied in regard to the case $s_1 \ll a$. The limiting values, given by equations (4.78, 4.79), differ from those derived for $a \ll s_2$ considerably, *cf.* equations (4.67, 4.68).

Contrary to the previous case of equations (4.29, 4.30) and (4.32, 4.33) the phase velocities expressed in equations (4.78, 4.79) are found to be strongly dependent on the elastic parameters of the background medium, including some of the properties of the reinforcing sheets.

6. CONCLUSION

The simple version of the effective stiffness theory expressed by the stochastic displacement equations of motion (1.10, 1.11) is employed for the determination of a deterministic system of equations (2.3) for mean wave solutions propagating in the laminated solid. Mean wave modes traveling parallel and perpendicular to the direction of the layering are considered, by assuming a wide-band flat spectrum process, see equations (4.22, 4.23), for the elastic parameters of the soft matrix material. Keller's perturbation method is modified by introducing the spectral densities of the random coefficients, see equations (4.24, 4.25), which makes it possible to use the Fourier transformed Green's dyadic of the background medium, thus simplifying the theory of the mean wave equations to a great extent. As a main result, the mean P- and S-wave modes propagating in x- or z-direction are found to be uncoupled, dispersive, yet non-attenuated waves. The dispersion relations are given in explicit form in equations (4.32 through 4.35) and (4.60, 4.61) and (4.76, 4.77) for those wave systems. The additional dispersion, which occurs in the mean waves, is found to be proportional to the variance of the random parameters and dependent on the band limit of the spectrum of these parameters. Through the appliance of the effective stiffness theory to the background medium, see [4], it has become evident that the theory is more suitable for describing waves propagating parallel to the layering. The waves traveling perpendicular to the layering of the random medium are found to be dispersive, *cf.* equations (4.34, 4.35), yet the phase velocities are independent of the elastic parameters of the stiff reinforcing sheets. However, the waves propagating in x-direction are dispersive, their phase velocities being strongly dependent on the elastic properties of the reinforcing sheets, *cf.* equations (4.78 through 4.80).

Acknowledgement—The author gratefully acknowledges support from a Max Kade Foundation Fellowship at the Technological Institute, Northwestern University, Evanston, Illinois, during the progress of this work. This work was supported in part by Grant AF-AFOSR-100-67. The author is indebted to Professors Jan. D. Achenbach and George Herrmann for their constant help and encouragement he received during the preparation of this paper.

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APPENDIX A

The averaged expressions (4.10) are determined from the two components

$$\alpha^{-4}F_1(x_1,z_1) = \left[R_{11}(z_1)\frac{\partial^2}{\partial x^2}+R_{12}(z_1)\frac{\partial^2}{\partial z^2}+R_{13}(z_1)\frac{\partial}{\partial z}\right]\bar{\bar{G}}_{11}(x_1,z_1)$$
$$+\left[R_{14}(z_1)\frac{\partial^2}{\partial x\,\partial z}+R_{15}(z_1)\frac{\partial^2}{\partial x^2}+R_{16}(z_1)\frac{\partial^2}{\partial z^2}+R_{17}(z_1)\frac{\partial}{\partial x}+R_{18}(z_1)\frac{\partial}{\partial z}\right]$$
$$\times\bar{\bar{G}}_{12}(x_1,z_1)+\left[R_{19}(z_1)\frac{\partial^2}{\partial x\,\partial z}+R_{20}(z_1)\frac{\partial}{\partial x}\right]\bar{\bar{G}}_{22}(x_1,z_1)$$

and

$$\alpha^{-4}F_2(x_1,z_1) = \left[R_{21}(z_1)\frac{\partial^2}{\partial x\,\partial z}+R_{22}(z_1)\frac{\partial}{\partial x}\right]\bar{\bar{G}}_{11}(x_1,z_1)$$
$$+\left[R_{23}(z_1)\frac{\partial^2}{\partial x\,\partial z}+R_{24}(z_1)\frac{\partial^2}{\partial x^2}+R_{25}(z_1)\frac{\partial^2}{\partial z^2}+R_{26}(z_1)\frac{\partial}{\partial x}+R_{27}(z_1)\frac{\partial}{\partial z}\right]$$
$$\times\bar{\bar{G}}_{21}(x_1,z_1)+\left[R_{28}(z_1)\frac{\partial^2}{\partial x^2}+R_{29}(z_1)\frac{\partial^2}{\partial z^2}+R_{30}(z_1)\frac{\partial}{\partial z}\right]\bar{\bar{G}}_{22}(x_1,z_1)$$

where

$R_{11} = a_{11}R(z_1)+a_{21}T(z_1)$	$R_{21} = (1+\bar{v})R_{12}$
$R_{12} = b_{11}R(z_1)+b_{21}T(z_1)$	$R_{22} = \bar{v}R_{13}$
$R_{13} = b_{11}\bar{T}(z_1)+b_{21}S(z_1)$	$R_{23} = (1+\bar{v})R_{16}$
$R_{14} = (1+\bar{v})R_{12}$	$R_{24} = R_{12}$
$R_{15} = A_{11}R(z_1)+A_{21}T(z_1)$	$R_{25} = R_{11}$
$R_{16} = B_{11}R(z_1)+B_{21}T(z_1)$	$R_{26} = \bar{v}R_{18}$
$R_{17} = R_{13}$	$R_{27} = a_{11}\bar{T}(z_1)+a_{21}S(z_1)$
$R_{18} = B_{11}\bar{T}(z_1)+B_{21}S(z_1)$	$R_{28} = R_{16}$
$R_{19} = (1+\bar{v})R_{16}$	$R_{29} = R_{15}$
$R_{20} = R_{18}$	$R_{30} = A_{11}\bar{T}(z_1)+A_{21}S(z_1)$

and

$$
\begin{aligned}
b_{11} &= -k^2\{A[1+(1+\bar{\nu})n_1^2]+Bn_1n_2(1+\bar{\nu})\} \\
b_{21} &= ik(An_2+Bn_1) \\
B_{11} &= -k^2\{An_1n_2(1+\bar{\nu})+B[1+(1+\bar{\nu})n_2^2]\} \\
B_{21} &= ik[A\bar{\nu}n_1+B(2+\bar{\nu})n_2] \\
a_{m1} &= (2+\bar{\nu})b_{m1}, \qquad A_{m1} = (2+\bar{\nu})B_{m1}, \qquad m = 1, 2.
\end{aligned}
$$

The correlation functions $R(z_1)$, T, $\bar{T}$, $S(z_1)$ are expressed in equations (4.7–4.9); $\bar{\bar{G}}_{ij}(x_1, z_1)$ is given formally in equation (3.7).

APPENDIX B

The integrals which determine the expressions (4.16, 4.17) are expressed in part by equations (4.19, 4.20) and from the following table

$$
\begin{aligned}
\mathrm{I}_3 &= b_{21}\alpha^4(2\pi)^{-1}\int_{-\infty}^{\infty} i(s+\zeta)G_{11}(\xi, s+\zeta)\,\Psi(s)\,\mathrm{d}s \\
\mathrm{I}_4 &= -(1+\bar{\nu})b_{11}\alpha^4\frac{1}{2\pi}\xi\int_{-\infty}^{\infty}(s+\zeta)G_{12}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s \\
\mathrm{I}_5 &= -A_{11}\alpha^4\xi^2(2\pi)^{-1}\int_{-\infty}^{\infty}G_{12}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s \\
\mathrm{I}_6 &= -B_{11}\alpha^4(2\pi)^{-1}\int_{-\infty}^{\infty}(s+\zeta)^2G_{12}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s \\
\mathrm{I}_7 &= b_{21}\alpha^4\frac{i}{2\pi}\xi\int_{-\infty}^{\infty}G_{12}(\xi, s+\zeta)\Psi(s)\,\mathrm{d}s \\
\mathrm{I}_8 &= B_{21}\alpha^4\frac{i}{2\pi}\int_{-\infty}^{\infty}(s+\zeta)G_{12}(\xi, s+\zeta)\Psi(s)\,\mathrm{d}s \\
\mathrm{I}_9 &= -(1+\bar{\nu})B_{11}\alpha^4\xi(2\pi)^{-1}\int_{-\infty}^{\infty}(s+\zeta)G_{22}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s \\
\mathrm{I}_{10} &= B_{21}\alpha^4\frac{i}{2\pi}\xi\int_{-\infty}^{\infty}G_{22}(\xi, s+\zeta)\Psi(s)\,\mathrm{d}s \\
\mathrm{II}_1 &= -(1+\bar{\nu})b_{11}\alpha^4\xi(2\pi)^{-1}\int_{-\infty}^{\infty}(s+\zeta)G_{11}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s \\
\mathrm{II}_2 &= \bar{\nu}b_{21}\alpha^4\frac{i}{2\pi}\xi\int_{-\infty}^{\infty}G_{11}(\xi, s+\zeta)\Psi(s)\,\mathrm{d}s \\
\mathrm{II}_3 &= -(1+\bar{\nu})\alpha^4B_{11}\xi(2\pi)^{-1}\int_{-\infty}^{\infty}(s+\zeta)G_{21}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s
\end{aligned}
$$

$$\mathrm{II}_4 = -b_{11}\alpha^4(2\pi)^{-1}\xi^2\int_{-\infty}^{\infty} G_{21}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s$$

$$\mathrm{II}_5 = -a_{11}\alpha^4(2\pi)^{-1}\int_{-\infty}^{\infty} (s+\zeta)^2 G_{21}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s$$

$$\mathrm{II}_6 = \bar{v}B_{21}\alpha^4\xi\frac{i}{2\pi}\int_{-\infty}^{\infty} G_{21}(\xi, s+\zeta)\Psi(s)\,\mathrm{d}s$$

$$\mathrm{II}_7 = a_{21}\alpha^4\frac{i}{2\pi}\int_{-\infty}^{\infty} (s+\zeta)G_{21}(\xi, s+\zeta)\Psi(s)\,\mathrm{d}s$$

$$\mathrm{II}_8 = -B_{11}\alpha^4(2\pi)^{-1}\xi^2\int_{-\infty}^{\infty} G_{22}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s$$

$$\mathrm{II}_9 = -A_{11}\alpha^4(2\pi)^{-1}\int_{-\infty}^{\infty} (s+\zeta)^2 G_{22}(\xi, s+\zeta)\Phi(s)\,\mathrm{d}s$$

$$\mathrm{II}_{10} = A_{21}\alpha^4\frac{i}{2\pi}\int_{-\infty}^{\infty} (s+\zeta)G_{22}(\xi, s+\zeta)\Psi(s)\,\mathrm{d}s$$

The coefficients b_{k1}, B_{k1}, a_{k1} and $A_{k1}, (k = 1, 2)$ are given in Appendix A and $G_{ij}(\xi, \zeta)$ are expressed by equations (3.3–3.6).

APPENDIX C

In terms of the basic integrals (4.55–4.58) the non-zero integrals which occur in the dispersion relations of mean P- and SV- waves propagating parallel to the layering are given by

$$\mathrm{I}_1 = \alpha^4(2+\bar{v})^2 A\frac{\Phi_0 k^4}{2\pi}(D_S^2\mathrm{IV}_0 - \bar{v}_1^2\mathrm{IV}_2)$$

$$\mathrm{I}_2 = \alpha^4(2+\bar{v})A\frac{\Phi_0 k^2}{2\pi}(D_S^2\mathrm{IV}_2 - \bar{v}_1^2\mathrm{IV}_4)$$

$$\mathrm{I}_4 = \alpha^4(1+\bar{v})(2+\bar{v})\bar{b}^2 A\frac{\Phi_0 k^4}{2\pi}\mathrm{IV}_2$$

$$\mathrm{I}_8 = -\alpha^4\bar{v}\bar{b}^2 A\frac{\Phi_0 k^2}{2\pi}\mathrm{IV}_4$$

$$\mathrm{I}_{10} = \alpha^4\bar{v}A\frac{\Phi_0 k^2}{2\pi}(D_P^2\mathrm{IV}_2 - \bar{v}_2^2\mathrm{IV}_4)$$

$$\mathrm{II}_2 = -\alpha^4\bar{v}B\frac{\Phi_0 k^2}{2\pi}(D_S^2\mathrm{IV}_2 - \bar{v}_1^2\mathrm{IV}_4)$$

$$\mathrm{II}_3 = \alpha^4(1+\bar{v})\bar{b}^2 B\frac{\Phi_0 k^4}{2\pi}\mathrm{IV}_2$$

$$\mathrm{II}_7 = -\alpha^4(2+\bar{v})\bar{b}^2 B\frac{\Phi_0 k^2}{2\pi}\mathrm{IV}_4$$

$$\mathrm{II}_8 = -\alpha^4 B\frac{\Phi_0 k^4}{2\pi}(D_P^2\mathrm{IV}_0 - \bar{v}_2^2\mathrm{IV}_2)$$

$$\mathrm{II}_9 = -\alpha^4(2+\bar{v})B\frac{\Phi_0 k^2}{2\pi}(D_P^2\mathrm{IV}_2 - \bar{v}_2^2\mathrm{IV}_4).$$

These expressions are considered through equations (4.49, 4.52) and the sums of I_i and II_i are given in equations (4.62, 4.63).

(*Received* 20 *August* 1968; *revised* 24 *January* 1969)

Абстракт—Даются стохастические уравнения в перемещениях для произвольных слоистых сред, пуетем использования простого варианта так называемых теорий "эффективной жесткости" слоистых сред. Процесс возмущения приводит к определяющим уравнениям, для среднего числа распространения волны. Обращается специальное внимание на несопряженные формы средних чисел волн, перпендиакулярных или параллельных к направлению слоя. Приводятся зависимости дисперсии для этих групп плоских волн.